

GLOBAL UNIQUENESS OF A CLASS OF MULTIDIMENSIONAL INVERSE PROBLEMS

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Uniqueness theorems for multidimensional inverse problems have at present been obtained mainly in classes of piecewise analytic functions and similar classes or locally (see [1]–[7], [15] and the literature cited there). Moreover, the technique of investigating these problems has, as a rule, depended in an essential way on the type of the differential equation. In this note a new method of investigating inverse problems is proposed that is based on weighted a priori estimates. This method makes it possible to consider in a unified way a broad class of inverse problems for those equations $Pu = f$ for which the solution of the Cauchy problem admits a Carleman estimate of the type considered in [8]–[11]. The theorems of §1 were proved by M. V. Klibanov and those of §2 by A. L. Buhgeĭm. They were obtained simultaneously and independently.

We consider the following inverse problems.

1.1. Determine functions $u(x, t)$, $a_{\alpha_0}(x)$, and $f(x)$ from the conditions

- (1) $u_t = Lu, \quad x \in R^n, \quad t > 0;$
- (2) $u(x, T) = F(x), \quad D^{\alpha_0} F(x) > 0, \quad T = \text{const}$
- (3) $u(x, 0) = f(x),$

where $L = \sum_{|\alpha| \leq 2} a_\alpha(x) D^\alpha$ is a uniformly elliptic operator of second order in R^n , and F and a_α , $\alpha \neq \alpha_0$, are given functions. Let T_0 be any number such that $T_0 > T > 0$, let $\Omega = R^n \times (0, T_0)$, let G be a domain in R^n , and let $H^{k+\beta}(R^n)$ and $H^{2k+\beta, k+\beta/2}(\Omega)$ (k is an integer, $\beta \in (0, 2)$) be the Hölder spaces [12].

THEOREM 1. *Suppose that in (1)–(3) $a_\alpha, f \in H^{2+\beta}(R^n)$, the function a_{α_0} is known for $x \in G$, and $a_\alpha \in C^\infty(G)$, $|\alpha| \leq 2$. Then there is not more than one vector-valued function (u, a_{α_0}, f) , $u \in H^{4+\beta, 2+\beta/2}(\Omega)$, satisfying conditions (1)–(3).*

The analyticity in t of the function u is used in the proof of Theorem 1 (see [13]). If the function a_{α_0} is not given in the domain G , then Theorem 1 is not true in general [14].

1.2. Determine functions $u(x, t)$ and $a_{\alpha_0}(x)$ from the conditions

- (4) $u_{tt} + Lu = 0, \quad x \in R^n, \quad t \in (0, T);$
- (5) $u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x);$

where f_1, f_2 , and a_α , $\alpha \neq \alpha_0$, are given functions.

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THEOREM 2. Suppose that in (4) and (5) the function a_{α_0} is known for $x \in G$ and $f_1, a_\alpha \in C^2(R^n)$, $|\alpha| \leq 2$, with $D^{\alpha_0} f_1(x) \neq 0$ for all $x \in R^n \setminus G$. Then there is not more than one pair of functions (u, a_{α_0}) , $u \in C^3(R^n \times [0, T])$, that is a solution of problem (4), (5).

1.3. Determine functions u and b from the conditions

$$(6) \quad \Delta_x u - \Delta_y u = \sum_{|\alpha| \leq 1} a_\alpha(x, y) D^\alpha u + b(x, y') f(x, y),$$

$$x \in \Omega_1 \subset R^n, \quad y \in \Omega_2 \subset R^m, \quad m, n \geq 1, \quad y' = (y_1, y_2, \dots, y_{m-1});$$

$$(7) \quad u(x, y', 0) = \psi_1(x, y'), \quad u_{y_m}(x, y', 0) = \psi_2(x, y');$$

$$(8) \quad u|_{x \in \partial \Omega_1} = F_1(x, y), \quad \frac{\partial u}{\partial n}|_{x \in \partial \Omega_1} = F_2(x, y).$$

Here $\Omega_1 = \{x \in R^n: |x| < r_1\}$ and $\Omega_2 = \{y \in R^m: |y| < r_2, y_m > 0\}$, $r_2 > r_1$; n is the unit normal to $\partial \Omega$; Δ is the Laplace operator (in the variables x or y respectively); and f, ψ_j, F_j , and a_α , $j = 1, 2$, $|\alpha| \leq 1$, are given functions. We set

$$B = \Omega_1 \times \Omega_2, \quad B' = B \cap \{y_m = 0\}, \quad Z_\delta = B' \times (0, \delta),$$

$$\omega_1 = \{(x, y) | |x| = r_1, |y| < r_1\}, \quad \omega_2 = \{(x, y) | |x|^2 - |y|^2 = 0\}.$$

Let $\Omega \subset B$ be the domain enclosed between the hyperplane $y_m = 0$ and the hyperplanes ω_1 and ω_2 ; let $\mathfrak{M} = C^2(\Omega) \cap C^1(\bar{\Omega})$.

THEOREM 3. Let $a_\alpha \in C(\bar{B})$, $|\alpha| \leq 1$, and suppose that $f \in C^3(Z_\delta)$ for some $\delta > 0$, and $f(x, y', 0) \neq 0$, $(x, y') \in \bar{B}'$. Then there is not more than one vector-valued function $(u, b) \in \mathfrak{M} \times C(\bar{B}')$ satisfying conditions (6)–(8).

2. In this section Ω is a bounded domain in R^n with a boundary of class C^∞ which is situated locally on one side of $\partial \Omega$.

2.1. We consider the two boundary value problems:

$$(9) \quad u_{rr}^j + L_j u^j = 0, \quad x \in \Omega, \quad t \in [0, T];$$

$$(10) \quad u^j(x, 0) = g(x), \quad u_r^j(x, 0) = 0, \quad x \in \Omega;$$

$$(11) \quad \frac{\partial u^j}{\partial n}|_{\Gamma \times [0, T]} = 0, \quad j = 1, 2; \quad \Gamma = \partial \Omega.$$

Here, as in §1, $L_j = \sum_{|\alpha| \leq 2} a_\alpha^j(x) D^\alpha$ is a uniform elliptic operator; $c_1 |\xi|^2 \leq \sum_{|\alpha| \leq 2} a_\alpha^j(x) \xi^\alpha \leq c_2 |\xi|^2$, $\xi \in R^n$, $c_1, c_2 > 0$; $D = (D_1, \dots, D_n)$, $D_j = \partial/\partial x_j$; $a_\alpha^j \in C^4(\bar{\Omega})$, $|\alpha| \leq 2$, and $a_\alpha^1 = a_\alpha^2$ for $\alpha \in A_1 = \{\alpha: |\alpha| \leq 2\} \setminus A_2$, where $A_2 \subseteq \{\alpha: |\alpha| \leq 2\}$ is some nonempty subset of multi-indices; $u^j \in C^4(\bar{\Omega} \times [0, T])$, and $g \in C^4(\Omega)$ is a vector-valued function of dimension $|A_2|$, where $|A_2|$ is the power of A_2 ; finally, n is the unit normal to Γ . We denote by $B(x)$ the matrix of order $|A_2| \times |A_2|$ with columns which are the vector-valued functions $D^\alpha g$, $\alpha \in A_2$.

THEOREM 4. Suppose that $\det B(x) \neq 0$ for all $x \in \bar{\Omega}$ and $u^1 = u^2$ for $(x, t) \in \Gamma \times [0, T]$. Then $a_\alpha^1 = a_\alpha^2$ for all α .

REMARK 1. If in (9) we replace u_{rr}^j by $-u_{rr}^j$ (the hyperbolic case), then for this problem there is a theorem analogous to Theorem 4 under the condition that the number $T = T(\Omega)$ is sufficiently large and the surface $\Gamma \times [0, T]$ and cylinders similar to it imbedded

in $\Omega \times [0, T]$ are strongly pseudoconvex in the sense of [8] with respect to the operator $D_t^2 - L_1$.

If in (9) we replace u_{tt}^j by $-u_t^j$ (the parabolic case) and condition (10) by $u^j(x, T') = g(x)$, $T' \in (0, T)$, then for this inverse problem there is also an analogue of Theorem 4.

2.2. Suppose that φ is a real function of class $C^\infty(\bar{\Omega})$ with $|\varphi(x)| + |\nabla\varphi(x)| \neq 0$, $x \in \bar{\Omega}$; let $\Omega_0 = \Omega \cap \{\varphi > 0\}$. We suppose that $\Gamma_0 = \partial\Omega_0 \setminus \{\varphi = 0\}$ and $\Sigma \subset \bar{\Omega}$ are hypersurfaces of class C^∞ .

We consider the problem of determining functions $u, f \in C^\infty(\bar{\Omega}_0)$ from the conditions

$$(12) \quad Pu = f, \quad Qf = g;$$

$$(13) \quad pu = (p_0u, p_1u, \dots, p_{m-1}u) = h_1, \quad qu = (q_0u, q_1u, \dots, q_{e-1}u) = h_2.$$

Here P and Q are linear differential operators of order m and l respectively with coefficients in $C^\infty(\bar{\Omega})$;

$$p_j u = \frac{\partial^j u}{\partial n^j} \Big|_{\Gamma_0}, \quad q_j u = \frac{\partial^j u}{\partial n^j} \Big|_{\Sigma},$$

n is the unit normal to Γ_0 or Σ respectively; and the function g and the vector-valued functions h_1 and h_2 are given.

We say that the operators P and Q *essentially commute* if the order of the operator $[P, Q] = PQ - QP$ is equal to $m + l - 2$. We assume that the following estimates hold for P and Q for all $\tau \geq \tau_0$:

$$(14) \quad \forall u \in C_0^\infty(\Omega_0) \quad \tau \|u\|_{\tau, m-1}^2 + \frac{a}{\tau} \|u\|_{\tau, m}^2 \leq C \|Pu\|_{\tau, 0}^2;$$

$$(15) \quad \forall u \in C_0^\infty(\Omega_0) \quad qu = 0 \quad \tau^\alpha \|u\|_{\tau, s+l-1}^2 \leq C_1 \|Qu\|_{\tau, s}^2,$$

$s = m - 1, m$, where the constants $C, C_1 > 0, a \geq 0$, and α do not depend on u or τ , and

$$\|u\|_{\tau, m}^2 = \sum_{|\alpha| \leq m} \int |D^\alpha u|^2 e^{2\tau\varphi} dx.$$

THEOREM 5. *Suppose that the operators P and Q essentially commute $\alpha > -1$ or $\alpha = 1$, and $8CC_1 \| [P, Q] \|_{L(W^{l+m-1, W^0})} < a$ ($W^s = W_2^s(\Omega_0)$ is the Sobolev space of order s). If the hypersurface Γ_0 is characteristic relative to Q , then the solution of problem (12), (13) under conditions (14) and (15) is unique.*

REMARK 2. In applications usually $l = 1$, $\Sigma = \Omega_0 \cap \{x_n = 0\}$, and the leading part of the operator Q is D_n . In this case for the validity of the estimate (15) with $\alpha = 0$ it suffices that $\varphi(x', x_n) \leq \varphi(x', 0)$, $x' = (x_1, \dots, x_{n-1})$. If $\varphi'_{x_n}(x', 0) = 0$ and $\varphi''_{x_n x_n} < 0$, then (15) holds for $\alpha = 1/2$. Without the condition $qu = 0$ the estimate (15) is not true.

Questions of the uniqueness and stability of a broad circle of inverse coefficient problems reduce to the problems considered in §§1.3 and 2.2. Theorems 1–4 essentially follow from Theorem 5. Theorem 4 and theorems similar to it (see Remark 1) can also be proved by the method of §1.2.

The operators P and Q in Theorem 5 may also be integrodifferential operators, and their coefficients may be matrices. The smoothness assumptions can be considerably relaxed. Theorem 5 has an analogue in anisotropic Sobolev spaces.

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